

## Quant Corner

$$h_t = \omega + \alpha \epsilon_{t-1}^2 + \beta h_{t-1}$$

# The Value of Volatility

**Robert Engle** won a Nobel prize for his ARCH model, which is now an essential tool for pricing derivatives such as variance swaps.

By Peter Carr

◀ Robert Engle, co-winner of the 2003 Nobel Memorial Prize in Economic Sciences, escapes from the academic world by taking to the rink as an ice dancer. After competing as an adult for almost 40 years, Engle placed second in the U.S. Figure Skating Association's adult championships in 1999 with his ice dancing partner, Wendy Buchi, and then retired from competitive skating.

Engle, 64, who's now a professor of finance at New York University's Stern School of Business, still goes to the rink three times a week. The enthusiasm and dedication to the sport that's required to excel in ice dancing also helped him develop and maintain a similar dedication to his research and teaching.

Engle received the Nobel prize in economics along with Clive Granger of the University of California, San Diego. When the award was announced, the Nobel committee cited Engle's "auto-regressive conditional heteroskedasticity" model, which analyzes random variables that have different variances from the mean, and its application to economic data with volatility that varies over time. The ARCH model has enabled derivatives analysts and other participants in financial markets to better understand time series of volatility.

Engle credits his father, a chemist, for sparking his interest in science. After completing a bachelor of science degree in physics in 1964 at Williams College in Williamstown, Massachusetts, Engle entered the graduate physics program at Cornell University in Ithaca, New York. Once at Cornell, he decided to study economics instead. "I decided I didn't really want to spend my life working on things that only 10 people in the world could understand," Engle says. "Economics was such a breath

of fresh air, because you're talking about problems that face everybody."

Engle took undergraduate courses in economics while finishing the requirements for his master's degree in physics, which he earned in 1966, and then entered Cornell's doctoral program in economics. He received his Ph.D. in '69.

Engle's first academic appointment was at Massachusetts Institute of Technology in Cambridge, where he taught economics until 1975. He then moved to the University of California, San Diego. Engle says he began to think about the idea of measuring volatility as a time series, meaning across historical periods, in the late '70s. At the time, the only way to quantify volatility was as a cross section, which uses one time period. Volatility is the key variable in forecasts of financial market conditions.

A POINT FORECAST is the average of all possible outcomes. The variance, which measures the spread of results, is the average of all of the squared deviations of each outcome from this mean. The larger the variance, the less certainty one has about the future, and the less relevant the point forecast becomes for financial decision making.

As an investor tries to peer into the future, the point forecast will, in general, change the further in time the investor looks, and the variance around that point forecast can only grow larger. The simplest financial models posit a linear relationship between variance and the horizon length. The random walk model of returns, which states that all future returns are independent of past results, has this property.

Financial econometricians, who apply statistics to finance, have known

## Robert Engle

### ARCH model for forecasting volatility

**Current job:** Professor of finance and Michael Armellino professor in the management of financial services at New York University's Stern School of Business

**Background:** Received a bachelor of science degree in physics from Williams College in 1964. Earned a master's degree in physics from Cornell University in 1966 and a Ph.D. in economics from Cornell in 1969. Was professor of economics at various levels at Massachusetts Institute of Technology from 1969 to '75. Joined the University of California, San Diego, where he taught until he moved to the Stern School in 1999.

**Personal:** Age 64. Born in Syracuse, New York. Has a son and a daughter.

for a long time that the relationship between variance and horizon length is more complicated than that predicted by the random walk hypothesis. For example, in a paper called "The Variation of Certain Speculative Prices" published in the *Journal of Business* in 1963, Benoit Mandelbrot observed that large changes tend to be followed by other large changes, while small changes are usually followed by other small changes. This phenomenon is now known as volatility clustering. Volatility is defined as the square root of the rate at which the variance of returns grows with the time horizon. When volatility clusters, the random walk hypothesis is refuted.

The ARCH model, which Engle described in 1982 in a paper called "Autoregressive Conditional Heteroskedasticity With Estimates of the Variance of U.K. Inflation" in the journal

## BLOOMBERG TOOLS

## Applying GARCH to Analyze Volatility

The GARCH Volatility (GRCH) function displays the results of a GARCH(1,1) stochastic, or random, volatility model as applied to a historical time series of the price of an asset. You can use GRCH to evaluate the volatility of bonds, stocks, currency exchange rates, fund returns or commodities. GRCH lets you search for patterns of discrepancy between the historical volatility—either standard historical volatility or volatility estimated from the GARCH model—and the implied volatility. Such discrepancies can offer opportunities for volatility trading. You can also use GARCH analysis to predict future variance levels.

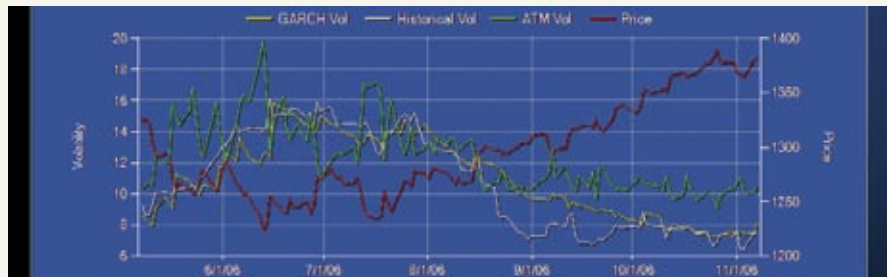
The GARCH(1,1) model applies a discrete-time, mean-reverting stochastic volatility process to the observed historical returns of the asset. GRCH uses a method of maximum likelihood to determine the speed and level of mean reversion and the volatility of volatility. The function uses these parameters to generate

a time series of volatility as implied by the Garch model.

For example, type SPX <Index> GRCH <Go> to analyze the volatility of the Standard & Poor's 500 Index. GRCH compares the volatility estimated by the GARCH(1,1) model to the standard historical volatility and also to the historical 30-day volatility implied by at-the-money options.

Click on the GARCH Var Swaps tab at the bottom of the screen to compare the Garch(1,1) model's variance forecast for different maturities with variance estimates from the options market. You can use these forecasts to price variance swaps.

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*Econometrica*, was the first hypothesis ever put forth that captured the phenomenon of volatility clustering.

Engle says the paper was inspired by Milton Friedman's 1976 Nobel lecture about the uncertainty of predicting inflation. "Friedman said that the reason we have recessions in business cycles is not inflation; it's the uncertainty of that inflation," Engle says. Engle's ARCH model solved the problem of finding a quantitative link between business cycles and inflation uncertainty.

The original ARCH model wasn't formulated using the continuous-time mathematics pioneered by Robert Merton, who won the Nobel prize in economics in 1997. ARCH is instead a discrete-time model, using a constant time step of length  $\Delta t$ .

To appreciate the context of Engle's contribution, it's necessary to step back to the earlier discrete-time models. In those models, the driving source of uncertainty was assumed to have independent and identically distributed (IID) increments, which ruled out the possibility of modeling volatility clustering.

Let  $\epsilon_t$  be an IID time series that's been de-measured, or has a mean of zero, and has constant variance  $\sigma_\epsilon^2$ . Suppose that the time series of returns is given by

$$r_t = \mu + \sum_{i=0}^{\infty} \gamma^i \epsilon_{t-i},$$

where  $\gamma \in [0, 1)$  measures the persistence of the uncertainty shocks. The returns  $\{r_t\}$  form a stationary time series, written as a one-sided moving average of the IID innovations  $\{\epsilon_t\}$ . The unconditional mean and variance of returns are both constant:

$$E[r_t] = \mu \text{ and } \text{Var}[r_t] = \sigma_\epsilon^2 \sum_{i=0}^{\infty} \gamma^{2i}.$$

Suppose we now condition the mean and variance on the information provided by the residuals up to time  $t - 1$ . The conditional mean is linear in equation 1 below:

$$E[r_t | \epsilon_{t-1}, \epsilon_{t-2}, \dots] = \mu + \sum_{i=1}^{\infty} \gamma^i \epsilon_{t-i}.$$

One might guess that the conditional variance depends on this information as well. On the contrary, the

conditional variance is constant in equation 2 below:

$$\begin{aligned} \text{Var}[r_t | \epsilon_{t-1}, \epsilon_{t-2}, \dots] &= \\ \text{Var}[\epsilon_t | \epsilon_{t-1}, \epsilon_{t-2}, \dots] &= \sigma_\epsilon^2. \end{aligned}$$

Recall that the standard model assumes that increments of the driving source of uncertainty are independent and identically distributed. By relaxing both assumptions in his 1982 paper, Engle accommodated volatility clustering. By retaining the defining property that the innovations  $\{\epsilon_t\}$  have a mean of zero, Engle preserved equation 1 so that the conditional mean was still linear in the data. Instead of assuming the increments are independent, Engle made the weaker assumption that they're serially uncorrelated. That allows the preservation of the first half of equation 2:

$$\begin{aligned} \text{Var}[r_t | \epsilon_{t-1}, \epsilon_{t-2}, \dots] &= \\ \text{Var}[\epsilon_t | \epsilon_{t-1}, \epsilon_{t-2}, \dots] &= \end{aligned}$$

Instead of having increments that are identically distributed, Engle assumed that the conditional variance depends linearly on all past squared shocks, as in equation 3 below:

$$\text{Var}[\epsilon_t | \epsilon_{t-1}, \epsilon_{t-2}, \dots] = \omega + \sum_{i=1}^{\infty} \alpha^i \epsilon_{t-i}^2,$$

where  $\omega > 0$  and  $\gamma \in [0, 1)$  affects the autocorrelation in squared shocks, which is their tendency to influence each other. As a result, the variance of returns shares this linear dependence:

$$\text{Var}[r_t | \epsilon_{t-1}, \epsilon_{t-2}, \dots] = \omega + \sum_{i=1}^{\infty} \alpha^i \epsilon_{t-i}^2.$$

Volatility clustering arises through similar dependence of adjacent variance rates on the same factors.

To ease the estimation of the parameters from time series, Engle assumed that the shocks all form a normal distribution, or a bell-shaped curve around the mean. Thus, the assumption on the shocks can be summarized as

$$\epsilon_t | \epsilon_{t-1}, \epsilon_{t-2}, \dots \sim N(0, h_t),$$

where  $h_t \equiv \text{Var}[\epsilon_t | \epsilon_{t-1}, \epsilon_{t-2}, \dots]$ .

In the absence of Engle's key equation 3, the conditional variance  $h_t$  is not directly observable at time  $t - 1$ . Engle's key idea is summarized in this equation, which relates the otherwise unobservable conditional variance  $h_t$  to the observable squared shocks in a simple way. The ARCH model is also consistent with distributions that have many outlying observations, provided one looks further than a single period into the future.

Academics who cite Engle's 1982 paper sometimes refer to the results as ARCH( $\infty$ ) to highlight the assumed dependence of the conditional variance on all past squared shocks. Since  $\alpha < 1$ , the weight on very old squared shocks is small. The required truncation of the infinite series hindered early applications of ARCH( $\infty$ ) to data. Tim Bollerslev, one of Engle's students, solved this problem. Bollerslev, who's now a professor at Duke University in Durham, North Carolina, published a paper called "Generalized Autoregressive Conditional Heteroskedasticity" in the *Journal of Econometrics* in 1986. That study proposed writing the conditional variance  $h_t$  alternatively as:

$$h_t = \omega + \alpha \epsilon_{t-1}^2 + \beta h_{t-1}.$$

This is the so-called GARCH specification; GARCH stands for *generalized ARCH*. Francis Diebold, a professor at the University of Pennsylvania in Philadelphia who published a survey of Engle's work in the *Scandinavian Journal of Economics*, has pointed out that GARCH is a specialized form of Engle's ARCH( $\infty$ ).

Many analysts apply the GARCH equation in their models for derivatives pricing because the value of many types of derivatives increases with higher volatility. Variance and volatility swaps are particularly well suited for valuing via the GARCH model. In a variance swap, two parties agree on a figure for realized variance over the life of the swap agreement. If the actual variance of the index at the end of the agreement exceeds the predefined value, the buyer of the swap receives the difference between the realized variance and the predefined value multiplied by the notional amount of the deal.

Volatility is the square root of variance

and serves as the underlying for a volatility swap. The ability to predict volatility or variance is a vital aspect of pricing both types of swaps. The timing and duration of volatility clusters can significantly change the value of a volatility swap. "There are plenty of hedge funds that are willing to take the other side of a position when somebody has used an inappropriate equation," Engle says.

Engle continues to develop new studies that apply to financial markets. He says one of his current projects involves analyzing volatility to determine whether a country's macroeconomic fundamentals determine its volatility.

Engle says he's also studying correlations to determine how they change over time and is working on a book called *Forecasting Correlation*. He says he strives to define a problem before he applies mathematical methods to analyze it. "Which problem is the right problem?" is a question Engle says he often asks.  $\blacktriangleright$

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